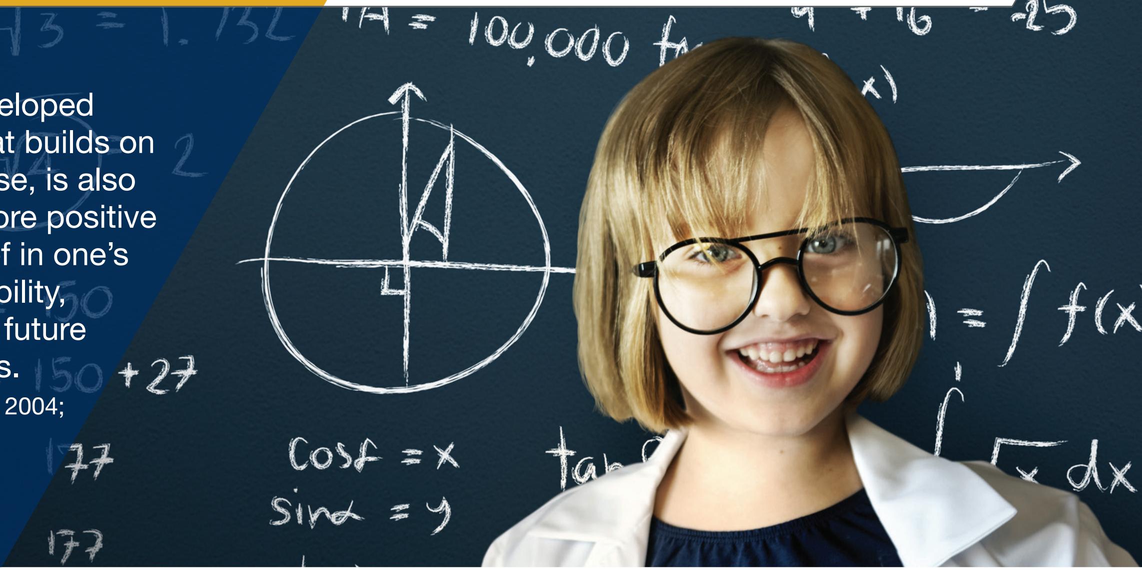
Acquiring a well-developed sense of number, that builds on intuitive number sense, is also associated with a more positive disposition and belief in one's own mathematical ability, and the likelihood of future study in mathematics.
(Beswick, Muir & McIntosh, 2004; Howden, 1989)



Big ideas in number

Building conceptual understanding of number and algebra

The Australian Curriculum: Mathematics incorporates the numeracy capabilities that all students need in their personal, work and civic life as well as providing fundamentals for building the professional applications of mathematics.

In the introduction to the General Capability: Numeracy, the Australian Curriculum states that:

Numeracy encompasses the knowledge, skills, behaviours and dispositions that students need to use mathematics in a wide range of situations. It involves students recognising and understanding the role of mathematics in the world and having the dispositions and capacities to use mathematical knowledge and skills purposefully. (ACARA: http://TLinSA.2.vu/GCnumeracy)

The Australian Curriculum: Mathematics is organised into three content strands: *number and algebra*, *measurement and geometry*, and *statistics and probability*. This series of *Best Advice* papers focuses on the number and algebra strand.

Number and algebra are developed together, as each enriches the study of the other. Students apply number sense and strategies for counting and representing numbers. They explore the magnitude and properties of numbers. They apply a range of strategies for computation and understand the connections between operations. (ACARA: http://TLinSA.2.vu/MathsStructure)

Number is a critical component of numeracy and this paper aims to draw out the key ideas and strategies needed to work confidently and flexibly with numbers in a variety of ways. Acquiring a well-developed sense of number, that builds on intuitive number sense, is also associated with a more positive disposition and belief in one's own mathematical ability, and the likelihood of future study in mathematics (Beswick et al, 2004; Howden, 1989).

Number sense

Number sense has been defined as *'a person's general understanding of number and operations along with the ability and inclination to use this understanding in flexible ways to make mathematical judgements and to develop useful and efficient strategies for managing numerical situations ... It results in a view of numbers as meaningful entities and the expectation mathematical manipulations and outcomes should make sense ... Those who use mathematics in this way continually utilise a variety of internal 'checks and balances' to judge reasonableness of numeric outcomes.'* (McIntosh, Reys, Reys, Bana & Farrell, 1997, p.3, in Siemon et al, 2015, p.202)

Throughout the mathematics curriculum there is a focus on developing increasingly sophisticated and refined mathematical proficiencies: understanding, fluency, reasoning, and problem-solving skills. These proficiencies enable students to respond to familiar and unfamiliar situations by employing mathematical strategies to make informed decisions and solve problems efficiently.

Learning to count with understanding is a crucial number skill. Other skills, however, such as perceiving subgroups, also need to develop alongside counting to provide a firm foundation for number sense. Different mental strategies can be prompted simply by presenting objects (such as stamps on a flashcard) in various arrangements. For example, showing six stamps in a cluster of four and a pair prompts the combination of 'four and two is six'. If four is not subitised (instantly recognised) by the learner then it may be seen as 'two and two and two is six'. The arrangement of two groups of three is much simpler

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and would not elicit the same response. So, different arrangements will prompt different strategies and these strategies will vary from person to person.

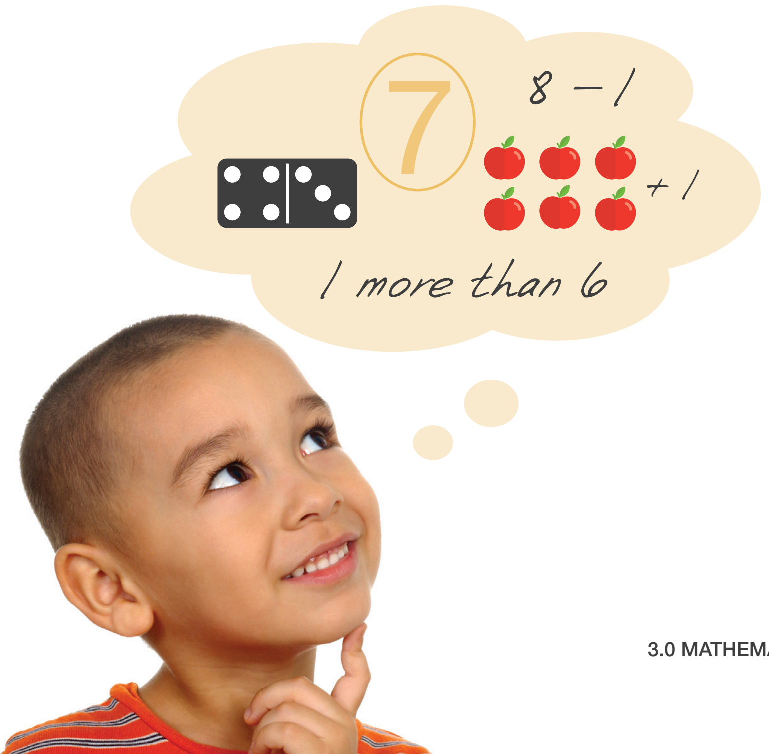
It is important from an early age to provide contexts that allow learners to use mathematical skills and knowledge in more than one way. Number is not merely a collection of arithmetic operations: adding, subtracting, dividing and multiplying. Number sense incorporates thinking and questioning the ways numbers are used in different contexts. This requires a deep understanding of the purpose and context for which numbers are used. This understanding is fostered as children develop deeper insights into how numbers work, including how they can be communicated, interpreted and manipulated to solve problems. The capacity to move flexibly between different number sets and representations requires a disposition to make sense of numbers in whatever context they occur.

An example of the importance of context is illustrated below:

The monkeys at the zoo ate 45 bananas in the morning and 37 bananas in the afternoon. How many banana skins were left on the enclosure floor at the end of the day?

When the above example was presented to two year 4 classes around a third of them subtracted to find the answer. Most said that they subtracted because the word 'left' was included in the question. Other explanations included subtracting because the bananas had been eaten and because the larger number appeared first (Siemon, 1993). In contrast, many of the students who added were able to paraphrase the problem in their own words.

An educator's main task is to scaffold this deep understanding. This is particularly important in relation to the six big ideas in number which help progress a well-developed sense of numeracy. These are outlined below and elaborated on in subsequent papers in this series.



Trusting the count

The term *trusting the count* was originally proposed by Willis (2002) to refer to the fact that learners may not believe that they would arrive at the same amount if they counted the same collection twice. Di Siemon expanded the term to also refer to a learner's capacity to access flexible, mental models for the numbers 0–10 (Siemon, Beswick, Brady, Clark, Faragher & Warren, 2015).

Children need a deep understanding of the numbers 0–10 by the end of their first year in school, not only in terms of what they represent but also how they might be reconfigured or viewed in relation to other numbers. Recognising the names and numerals is not enough. Children need flexible mental models for each of the numbers.

'*Young children spontaneously use the ability to recognize and discriminate small numbers of objects*' (Klein & Starkey in Clements, 1999). This intuitive ability to 'instantly see how many' is called subitising from a Latin word meaning suddenly. Two types of subitising have been identified: perceptual and conceptual (Clements, 1999). Perceptual subitising is the immediate recognition of small collections up to 4 or 5, whereas conceptual subitising involves seeing a collection of say 7, and recognise 7 instantly in terms of its subitised parts, for instance 'I see a four and a three'. In other words, it is the ability to see numbers as composite units. Both perceptual and conceptual subitising are foundational to the development of number sense. An intuitive, age-appropriate sense of number precedes, and involves more than, rote counting.

Subitising is a key indicator of the extent to which children have developed mental models for the numbers 0–10. So, when a child reads, hears or writes seven they can imagine what a collection might look like and how it relates to other numbers, for example, seven is 1 more than 6; 3 and 4; 7 dots on a domino; double 3 plus 1 etc.

Learners trust the count when they have access to a range of mental models for each of the numbers to ten without having to count by ones.

Learners who *trust the count*:

- **can represent the numbers 0–10 and beyond in multiple ways and in a range of contexts**
- **can recognise collections of numbers without counting one by one**
- **use comparative language related to collections and quantities.**

Place value

'Ten of these is one of those' is a key pattern underpinning initial place value understanding. *Place value* underpins fluency in arithmetic calculations and supports ongoing learning with larger numbers and decimals. Young learners can usually 'read' two-digit numbers before they actually understand that the placement of a digit in a number impacts its value. Learners may therefore be able to read 72 and 27 and state which one is larger/smaller, without knowing why the numbers have different values. An overemphasis on the mathematical recording of two-digit numbers undermines a learner's capacity for developing their mental and written computation strategies, particularly as our English number naming system does not support the place value pattern (eg eleven, twelve, the 'teens' as in thirteen, fourteen). Furthermore, many learners may consider 'twenty' as the name and '20' as the numeral for a collection of 20 objects. They are not necessarily connecting the '2' in '20' as 2 tens.

Place value is difficult to teach and learn and misunderstandings are often masked by the successful performance of superficial tasks such as counting by 1s on a 0–99 chart. The structure of the base 10 number system is essentially multiplicative, involving different sized groups that are powers of ten. Misconceptions develop when place value is introduced to learners before they have demonstrated that the numbers 2 to 10 can be used as countable units. Counting large collections efficiently, as well as comparing, ordering, counting forwards and backwards and understanding the relative size of numbers confidently in place value units, is a key indicator of the extent to which learners have developed a sound basis for place value.

By the end of year 2, students need a deep understanding of the *place value* pattern, eg 10 of these is one of those, to support efficient ways of working with two-digit numbers and beyond.

Learners who have an understanding of *place value*:

- **can make connections between amounts and a variety of representations**
- **can model (eg using concrete materials), name and record two-digit numbers; then three-digit numbers, plus explain place value to their peers or the teacher**
- **when talking about practical everyday problems, demonstrate their knowledge of place value by comparing, ordering, counting and renaming.**

Multiplicative thinking

Multiplicative thinking has been identified as one of the most important and also most difficult mathematical concepts for learners to develop (Siemon, Virgona & Cornielle, 2001) and it involves thinking about multiplication in a number of different ways. Multiplicative thinking is a prerequisite for the understanding of rational number and proportional reasoning: fractions, decimals, ratio, percent and proportion (Siemon, Izard, Breed & Virgona, 2006). Successfully transitioning from additive to multiplicative thinking is necessary before learners can fully understand proportional reasoning and algebra in high school mathematics.

Multiplicative thinking occurs when students can hold the number in each group (multiplicand), the number of groups (multiplier) and the total amount (product), in their minds at the same time (Jacob & Willis, 2001). Being able to picture multiplication as an array, region, or area model is important, as this helps learners to recognise and use equal groups in rows and columns, and teaches them that a collection can be rearranged whilst the quantity stays the same (Norbury, 2002). This is needed to support representation of fractions and decimals, multiplication and division of larger whole numbers, decimals and algebra. An awareness of the 'for each' idea, or Cartesian product, is also needed at this level of schooling to support work in statistics and probability, measurement and further work in proportional reasoning.

By the end of year 4, learners need to be able to think about multiplication in a range of ways so that they see when multiplication is required and how it relates to division.

Learners who have an understanding of *multiplicative thinking*:

- **use the language of multiplicative thinking (eg *for each* and *times as many*)**
- **can work with a constant number of groups, eg, 3 sixes, think, double six, 12 and 1 more six, 18**
- **can work flexibly with concepts, strategies and representations of multiplication (and division) in a wide range of contexts**
- **x a variety of representations, including words, arrays, regions, area models and tree diagrams to show multiplicative thinking when problem solving.**

Partitioning

Partitioning is the process of dividing an object or objects into equal groups or equal parts. The idea that a collection or quantity can be expressed in terms of its parts is fundamental in developing a strong sense of number. When working with fractions these parts must be equal. Fractions are a difficult concept for learners to grasp as there are so many different uses for fractions and so many different contexts in which they appear. A large proportion of students in years 5–9 experience difficulty with fractions, decimals and percentages (Siemon, Breed, Dole, Izard & Virgona, 2006).

The teaching and learning of fractions, particularly in the middle years and beyond, is a key aspect of numeracy as it underpins the ability to compare multiplicative relationships between quantities used in everyday life. Research has indicated that 90% of the most commonly encountered texts in households require knowledge of fractions, decimals, percent, ratio and proportion.

The extent to which students can construct their own fraction models and move fluidly between representations appropriate to context is a key indicator of the depth of their understanding.

Learners who have an understanding of *partitioning*:

- **can make their own fraction models and representations**
- **discuss and collaborate in small groups to purposefully partition wholes**
- **can describe, record, explain and justify their thinking when comparing, ordering, sequencing and renaming fractions.**

Proportional reasoning

Being able to think and reason proportionally is an essential element for understanding and applying mathematics. People use *proportional reasoning* to calculate best buys; to perform measurement or monetary currency conversions; to adjust recipes; and to work with drawings and plans.

The essence of proportional reasoning is the consideration of the relationship between two quantities. Proportional reasoning involves learners thinking multiplicatively, rather than additively, about the numbers involved (Behr, Harel, Post & Lesh, 1992). For example, instead of just describing a number or quantity as smaller than or bigger than, learners use terms such as double, half or three-times greater.

Learners are using proportional reasoning when they understand that a group of children that grows from 3 to 9 is a more significant change than a group of children that grows from 100 to 150, since the number tripled in the first example, but only grew by half (not even doubled) in the second example.

It is estimated that '*over 90% of students who enter high school cannot reason well enough to learn mathematics and science with understanding and are unprepared for real applications in statistics, biology, geography or physics …*' (Lamon in Ontario Ministry of Education, 2012, p.4)

Learners who have an understanding of *proportional reasoning*:

- **can use concrete materials to demonstrate their understanding of proportionality**
- **when required to reason proportionally, can use multiplicative (rather than additive) thinking**
- **can use precise terms, such as three-quarters, when describing number, quantity and relationships.**

Generalising

'*Generalising is the heartbeat of mathematics*' (Mason, 1996, p.1): it begins when learners start to notice patterns and spot aspects that stay the same when other aspects change. Identifying pattern and structure in mathematics is at the heart of mathematics. When learners are faced with a mathematical problem they are often required to identify a pattern and then give a general rule for what they see.

This shifts the focus in classroom mathematics from finding one correct numerical answer to recognising and constructing patterns and generalising mathematical relationships (important for algebraic work).

By exploring complex problems and finding multiple possible solutions, learners are encouraged to use reasoning about what they have learnt for one situation and transfer it to another in order to find a solution that will *always* work—the essence of a generalisation.

It is not enough though to just identify a pattern: learners need to be able to explain why the pattern works. This then leads to learners being able to generalise: being able to explain and apply the general rule in other instances.

Learners who have an understanding of *generalising*:

- **notice and explain patterns and structures in mathematics**
- **can work with equivalence and algebraic text**
- **construct patterns and recognise relationships**
- **understand when to transfer reasoning from one situation to another.**

How can leaders support their staff?

Some teachers may not possess confidence in their own understanding of number or the pedagogical repertoire needed to help students develop conceptual understanding. Creating a safe environment for rigorous learning, in which teachers can explore and confront their own misconceptions, is a very important starting point for growth. While this paper draws out the big ideas in number for leaders, it could be used as a starting point for teacher discussion and be supplemented with each of the individual papers on the big ideas that follow in this series.

Additionally, leaders could explore with staff Peter Sullivan's six key principles for effective teaching of mathematics (2011, 2014) to address the pedagogical content knowledge required for number and algebra. The principles are: articulating goals; making connections; fostering engagement; differentiating challenge; structuring lessons; and promoting fluency and transfer. The principles can be applied to the teaching and learning of number and algebra.

Furthermore, leaders could review the paper 'Eight effective practices that develop numeracy B–18 with question prompts' (DECD, 2016), which draws on the work of Sullivan (2011), Clark & Clark (2004) and the National Council of Teachers of Mathematics (NCTM, 2017). The eight effective practices are: fostering engagement; identifying learning goals; facilitating meaningful collaboration and dialogue; making connections; providing challenge; collecting and responding to evidence; building fluency from conceptual understanding; and using digital technology to connect.

When leaders support educators to examine their teaching of mathematics, the following three questions may be beneficial:

1. How often do learners experience these practices, in relationship to number and algebra, and how do you know?
2. Which of these practices are strengths and which ones need to be developed further?
3. Which practices are priorities to target in order to shift pedagogical practices and achieve site targets and priorities?

Leaders may also consider designing more specific questions in relationship to number and algebra, such as:

MAKING CONNECTIONS
In what ways are learners supported to make connections in number and algebra across mathematics and to their everyday lives?

BUILDING FLUENCY FROM CONCEPTUAL UNDERSTANDING
How do educators at your site provide a balance between direct instruction of number and exploring a number concept?

COLLECTING AND RESPONDING TO EVIDENCE
How do educators at your site use evidence of learners' thinking in number and algebra to track, monitor and respond to learner achievement?

Further resources

The big ideas in number are discussed in further detail in the following mathematics papers:

- 3.1 Trusting the count
- 3.2 Place value
- 3.3 Multiplicative thinking
- 3.4 Partitioning
- 3.5 Proportional reasoning
- 3.6 Generalising.

All papers in this series are based on the work of Dianne Siemon, Professor of Mathematics Education at RMIT and a key text (Siemon et al, 2015).

http://TLinSA.2.vu/BestAdviceNum

Further reading

ACER PAT Teaching Resources Centre houses relevant concept builders related to the number and algebra strand.

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Ontario Ministry of Education (2012) Paying attention to proportional reasoning: Support document for paying attention to mathematical education, Queen's Printer for Ontario, available at http://TLinSA.2.vu/OntarioProportionReason (accessed March 2020)

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